

A Fast Adaptive Wavelet scheme in RBF Collocation for nearly singular potential PDEs

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Abstract: We present a wavelet based adaptive scheme and investigate the efficiency of this scheme for solving nearly singular potential PDEs. Multiresolution wavelet analysis (MRWA) provides a firm mathematical foundation by projecting the solution of PDE onto a nested sequence of approximation spaces. The wavelet coefficients then were used as an estimation of the sensible regions for node adaptation. The proposed adaptation scheme requires negligible calculation time due to the existence of the fast Discrete Wavelet Transform (DWT). Certain aspects of the proposed adaptive scheme are discussed through numerical examples. It has been shown that the proposed adaptive scheme can detect the singularities both in the domain and near the boundaries. Moreover, the proposed adaptive scheme can be utilized for capturing the regions with high gradient both in the solution and its spatial derivatives. Due to the simplicity of the proposed method, it can be efficiently applied to large scale nearly singular engineering problems.

Keywords: RBF collocation, Wavelet decomposition, Multiresolution analysis, Adaptive distribution, Potential problem, Multiquadrics

1 Introduction

Radial basis functions were introduced by Franke (1982) to mathematical community; these are effective tools in the numerical solution of linear and nonlinear PDEs. In addition to a firm mathematical foundation provided of RBF methods, see Micchelli (1986), and Schaback and Wendland (2000), RBFs are well known for their accuracy and spectral convergence if the solution is sufficiently smooth and

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has a regular behavior. However, singularities and localized features often emerge in many physical problems like crack front stress concentration, shock wave formation, temperature concentration and etc. Nonlinear hyperbolic PDEs can develop true mathematical discontinuities, and the proper procedure requires enriching the solution space to contain both continuous and discontinuous functions; in multi-dimensional problems, the discontinuous solutions are products of the Heaviside function in the normal propagation direction and piece-wise continuous functions in the tangential directions, see Kansa, Aldredge, Ling (2008). Bernal and Kinde-lan (2008) enriched the solution space of RBFs with the first few terms of the Motz boundary singularity to achieve rapid convergence for the injection of molten plastic in molds. This paper is restricted to the treatment of continuous, but nearly singular problems that are also notoriously difficult.

Good representations of such nearly singular phenomena demand the use of non-uniform node distributions that adapt to the changes in the sharp transition region. One can track the position of the near singularity and increase the node density in that region. Such strategies are often based on knowledge of the solution itself, on empirical data or on front tracking adaptive scheme. In an automatic adaptive scheme, more nodes are automatically added on those parts of the domain with more detail and simultaneously a sufficient number of nodes is kept in the smooth regions.

A number of papers have been published in the last several years describing the adaptive strategy in RBF solution of PDEs. Schaback and Wendland (2000), Hon, Schaback and Zhou (2003) developed an adaptive scheme based on the greedy algorithm and achieved a linear convergence rate in interpolation and collocation problems. Hon (1999) proposed an adaptive multiquadric scheme using an "a posterior" indicator based on the weak formulation of the governing equation to detect sharp transition regions and add more nodes where deemed necessary. Ling and Trummer (2006) modified the Hon's indicator to make it suitable for transformed boundary value problems. Sarra (2005) developed an adaptive RBF distribution based on simple equidistribution of an arclength algorithm and successfully applied it to the solution of nearly singular and time dependent Burger's and Advection equation in 1D. A dynamic adaptive scheme was proposed by Wu (2004,2005) for time-dependent PDEs. Bozzini, Lemardizzo, and Schaback (2002) have formulated an adaptive RBFs interpolation based on combining B-spline techniques with a scaled MQ. An adaptive algorithm with local TPS-RBFs interpolation was developed by Behrens and Iske (2002), Behrens, Iske, and Kaser (2003) successfully applied it, in a semi-Lagrangian context, to linear evolutionary PDEs. The method uses a local interpolation to evaluate an error indicator and to detect the regions where the approximate solution requires more accuracy. Gomez, Casanova, and

Gomez (2006) combined the former adaptive scheme with a quad-tree type algorithm in two dimensions. More recently, Driscoll and Heryudono (2007) presented an adaptive RBF scheme for time independent problems, specifically in interpolation and also linear and non-linear boundary value problems; their indicator used to refine the distribution in the region with a highly localized feature is based on the residual sub-sampling technique.

Even though all these adaptive strategies are mainly based on utilizing an indicator to detect the localized regions and adaptively allocate more nodes to those parts of the domain, they differ on practical aspects such as the types of the indicator used or the node refinement criteria. Many adaptive strategies mentioned above are driven by a front tracking scheme that utilizes a posterior error indicator to detect the regions that require refinement, see Lee, Im, Jung, Kim and Kim (2007) and Iske and Käser (2005).

One of the biggest issues in adaptive mesh refinement based on a posterior error indicator is that these adaptive schemes often dramatically penalize the simulation speed. So, there is still a need for an efficient, fast and fully adaptive method for solving nearly singular problems. That is where Multi Resolution Wavelet Analysis (MRWA) plays a role. Recently, the wavelet analysis has been developed as a potential adaptive approach for the construction of the optimum adaptive node distribution in nearly singular problems, see Cruz, Mendez, and Magallhaes (2001), Mehra and Kevlahan (2008), De Marchi, Franze, Baravelli, and Speciale (2006), Basilyev and Kevlajan (2005). The mathematical foundation of the algorithm is the MRWA that provides a firm mathematical foundation by projecting the solution of PDE onto a nested sequence of approximation spaces and examines the solution at different levels of resolution.

In recent years some attempts have been made to relate the RBFs with wavelets. The introduction of wavelets to RBFs analysis dates back to Micchelli, Rabut, and Utreras (1991), Buhmann (1995). Chui, Stoeckler, and Ward (1996) who have shown RBFs are wavelets that do not have orthogonality properties, i.e. they are prewavelets. Fasshauer and Schumaker (1998) summarized some wavelets using spherical RBFs. Buhmann and Micchelli (1992) and Chui, Ward, Jetter, and Stoeckler (1996) have shown that RBFs are prewavelets with dilatational, rotational and translational properties and are very good for detecting near singularities. For MQ-RBF, the term $\|\mathbf{x} - \mathbf{x}_j\|$ behaves as the wavelet translator, and the shape parameter c_j behaves as the dilator (scale) parameter. The non-orthogonality of RBF pre-wavelets are discussed by Micchelli, Rabut, and Utreras (1991) and Chen (2001) presented the orthonormal RBF wavelet series and transforms by using the nonsingular general solution and singular fundamental solution of the differential operator. The methodology presented by Chen (2001) can be generalized to RBF

wavelets by means of orthogonal convolution kernel function of various integral operators. However, to the best of our knowledge, the application of a wavelet based adaptive scheme in RBF analysis is still absent from the literature. Very recently, Libre, Emdadi, Kansa, Shekarchi, and Rahimian (2009) introduced an adaptive scheme based on MRWA decomposition for interpolation problems. Certain aspects of an adaptive wavelet scheme in MQ-RBF approximations have been discussed, and it was demonstrated that the adaptive prewavelet scheme can be fairly used for the detection of a boundary or an internal near singularity in interpolation problems.

The main objective of the present work is to develop the modified adaptive wavelet scheme presented in Libre, Emdadi, Kansa, Shekarchi, and Rahimian (2009) to solve nearly singular potential PDEs. The main question is how we can utilize the adaptive wavelet scheme for the solution of potential time independent PDEs and how efficient is the method? The rest of this paper is organized as follows. A short description of RBF collocation scheme for potential PDEs is presented in section 2. The adaptive wavelet scheme is briefly reviewed in section 3. The efficiency of the adaptive wavelet scheme for the solution of nearly singular potential problems is illustrated through several numerical examples in section 4. Finally, Conclusions are drawn in section 5.

2 RBF collocation method

Let us consider the following linear second order PDE, given by

$$L(u) = A(x,y) \frac{\partial^2 u}{\partial x^2} + 2B(x,y) \frac{\partial^2 u}{\partial x \partial y} + C(x,y) \frac{\partial^2 u}{\partial y^2} + D(x,y)u - f(x,y) = 0 \text{ in } \Omega \quad (1)$$

Together with the Neumann condition on natural boundary ($\partial\Omega_t$) and Dirichlet condition on essential boundary ($\partial\Omega_u$)

$$g(u) = \mathbf{n}^T \cdot \nabla u - g_n^* = 0 \text{ on } \partial\Omega_t \quad (2a)$$

$$u - u^* = 0 \text{ on } \partial\Omega_u \quad (2b)$$

Generally, the coefficients A, B, C and D may all depend upon the coordinates, x, y. However, if A=C=1, B=0 and C=k² then the PDE, Eq(1), will result in the well known isotropic Helmholtz equation in a two-dimensional domain, $\Omega \subset \mathbb{R}^2$ that is defined as

$$\Delta u(x,y) + k^2 u(x,y) = f(x,y), \quad (x,y) \in \Omega \quad (3)$$

where k is the wave number when it is real and positive and $f(x,y)$ represents a harmonic source. When $k=0$ the equation is known as the Poisson equation and when

both $k=0$ and $f(x,y)=0$ the well-known Laplace equation results. Helmholtz type equations arise in a variety of important physical applications; see Boa, Wei, and Zhao (2004), especially in acoustic and electromagnetic wave propagation. The convergence of the numerical solution of the Helmholtz equation depends significantly on the singular behavior of the solution. Generally speaking, the nearly singular regions should be analyzed at a high level of resolution. Consequently, the node spacing must be sufficiently refined to capture the near singularity. Very fine node spacing is required in the nearly singular problems, increases the total degree of freedom, the memory required for the data storing and the CPU time needed for the solution procedure. Moreover, the numerical instability arising from ill-conditioning phenomenon is more likely to occur in the nearly singular problems with more degrees of freedom. Therefore, alternative adaptive methods are urgently needed to attack the problem with nearly singular features. Treatment schemes encounter with near singularity problems in Helmholtz type equations are discussed in a series of papers, for example, see Marin, Lesnic, and Mantic (2004), Chen, Kuhn, Li, and Mishuris (2003), Qian, Han and Atluri (2004) and Marin (2008).

We now briefly review the RBF direct collocation method which will be used for the numerical solution of governing Helmholtz type PDEs. The RBF approximation is initially employed for the meshfree approximation of the solution of PDEs. The problem domain Ω is first discretized into a set of N_d nodes on the domain, N_u nodes on the essential boundaries Ω_u and N_t nodes on natural boundaries Ω_t . The solution $u(x)$ of a PDE and its derivatives are then approximated in 2D domain by:

$$u(x,y) = \sum_{i=1}^N \alpha_i \phi_i(x,y), \quad x,y \in \Omega \tag{4a}$$

$$\frac{\partial u}{\partial x} = \sum_{i=1}^N \alpha_i \frac{\partial \phi_i}{\partial x}, \quad \frac{\partial^2 u}{\partial x^2} = \sum_{i=1}^N \alpha_i \frac{\partial^2 \phi_i}{\partial x^2} \tag{4b}$$

$$\frac{\partial u}{\partial y} = \sum_{i=1}^N \alpha_i \frac{\partial \phi_i}{\partial y}, \quad \frac{\partial^2 u}{\partial y^2} = \sum_{i=1}^N \alpha_i \frac{\partial^2 \phi_i}{\partial y^2} \tag{4c}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \sum_{i=1}^N \alpha_i \frac{\partial^2 \phi_i}{\partial x \partial y} \tag{4d}$$

where $N=N_d+N_u+N_t$ is the total number of collocation nodes, Φ_i are the shape functions and α_i are their associated unknown coefficients. Multiquadric (MQ) $\phi_j = (r_j^2 + c_j^2)^{1/2}$, the Gaussian $\phi_j = \exp(-(r_j/c_j)^2)$, and the thin plate spline (TPS) $\phi_j = r_j^2 \log(r_j)$ are the most widely used global radial basis functions.

The RBF collocation method, as introduced by Kansa (1990a, 1990b), is formulated by introducing the above approximations of the solution and its derivatives

into the strong form of the governing equation and the corresponding boundary conditions. In this method, the governing PDE will be imposed at all N_d domain nodes, the Neumann condition at N_t nodes on the essential boundaries and Dirichlet condition at N_u nodes on Dirichlet boundaries. In general, the set of the collocation nodes can be different from the set of approximation nodes. However, for the sake of simplicity, collocation nodes are usually the same as the approximation nodes. The discretization procedure for general form of Helmholtz equation results in an $N \times N$ set of linear equations as follows:

$$\begin{bmatrix} \partial^2 \phi_i / \partial x^2 + \partial^2 \phi_i / \partial y^2 + k^2 \phi_i \\ n_1 \partial \phi_i / \partial x + n_2 \partial \phi_i / \partial y \\ \phi_i \end{bmatrix} \{\alpha_i\} = \begin{Bmatrix} f_i \\ g_i^* \\ u_i^* \end{Bmatrix} \quad (5)$$

We solve the $N \times N$ linear algebraic system for the unknown coefficients α_i and obtain the approximate solution and its derivatives at any point in the domain.

3 Adaptive wavelet scheme

The concept of wavelet analysis was introduced in applied mathematics by the end of the 1980s by Daubechies (1988) and Mallat (1989) and recently there is a growing interest in developing wavelet-based numerical algorithms in both the uniform and adaptive node distribution schemes for the solution of PDEs, see Cruz, Mendez, and Magallhaes (2001), Mehra and Kevlahan (2008), De Marchi, Franze, Baravelli, and Speciale (2006), Basilyev and Kevlajan (2005), Mitra and Gopalakrishnan (2006) and Xiang, Chen, Yang and He (2008). The wavelet based adaptation procedure which yields in compressed node distribution is almost the same as the well-known wavelet image compression method, for example, see Jun (2007).

The mathematical foundation of the adaptive wavelet algorithm is multi-resolution wavelet analysis, (MRWA). The MRWA projects a complicated function into a nested sequence of approximation subspaces $\{V_{j+1}\}_{j \in \mathbb{Z}}$, $V_j \subset V_{j+1}$ and establishes a set of scaling function coefficients a_{jk} and a set of wavelet coefficients d_{jk} , structured over different levels of resolution. Each of these subspaces $\{V_{j+1}\}_{j \in \mathbb{Z}}$, can be decomposed into an approximation space $\{V_j\}_{j \in \mathbb{Z}}$ and its orthogonal complement detail space $\{W_j\}_{j \in \mathbb{Z}}$. The space $L^2(\mathbb{R})$ can be expanded as an approximation space plus a sum of detail spaces, i.e. $L^2(\mathbb{R}) = V_{j_0} + \sum_{j=j_0}^{\infty} W_j$. The solution of PDEs can be expanded into the sum of its coarsest approximation u_{j_0} and series of additional detail functions, g_j .

$$u = u_{j_0} + \sum_{j=j_0}^{\infty} g_j = u_{j_0} + \sum_{j=j_0}^{\infty} \sum_k d_{j,k} \Psi_{j,k} \quad (6)$$

where $\psi_{j,k}$ are the bases of the detail space $\{W_j\}_{j \in \mathbb{Z}}$. In the MRWA, one can analyze the highly localized regions of a function at high levels of resolution and at the same time uses a low level of resolution for analyzing the function in flat regions. The key ingredient of MRWA is the existence of the fast Discrete Wavelet Transform (DWT), see Mallat (1989), that provides a simple means of transforming data from one level of resolution, j , to the next coarser level of resolution, $j - 1$.

$$a_{j-1,k} = \sum_l h_{2k-l} a_{j,l} \tag{7a}$$

$$d_{j-1,k} = \sum_l g_{2k-l} a_{j,l} \tag{7b}$$

Each scaling function coefficient a_{jk} and wavelet coefficient d_{jk} is associated to a certain node in a certain resolution level. The basic idea of the adaptive wavelet scheme is the fact that the wavelet coefficients involved in the low resolution level describe the smooth feature of the function while the wavelet coefficients at the highest level are associated with the highly localized feature. The high values of wavelet coefficients indicate an important fluctuation between the current level and the next coarser level of resolution. It is then evident how this concept can be applied in adaptive node distribution for the function with a highly localized phenomenon. Specific wavelet coefficients that associate a certain node in the domain can be appropriately identified or rejected, so that superfluous details are removed from the smooth regions. After applying the adaptation procedure, the distribution contains only the essentials nodes and this set tends to be the nearly optimal node distribution.

In the adaptive wavelet scheme, the solution, u , is decomposed into two parts u^1 , u^2 so that u^1 contains those terms whose wavelet coefficients amplitudes that are above some prescribed threshold ε and u^2 consists of those terms whose wavelet coefficients that are below the prescribed threshold ε .

$$u(x) = u^1(x) + u^2(x) \tag{8}$$

$$u^1(x) = \sum_k a_{j_0,k} \phi_{j_0,k} + \sum_{j=j_0} \sum_k d_{j,k} \psi_{j,k} \quad d_{j,k} \geq \varepsilon \tag{9a}$$

$$u^2(x) = \sum_{j=j_0} \sum_k d_{j,k} \psi_{j,k} \quad d_{j,k} < \varepsilon \tag{9b}$$

The low values of wavelet coefficients indicate that the resolution level can be appropriately decreased, avoiding unnecessary dense node distributions. That means the u^2 and the corresponding node distribution may be neglected from the solution

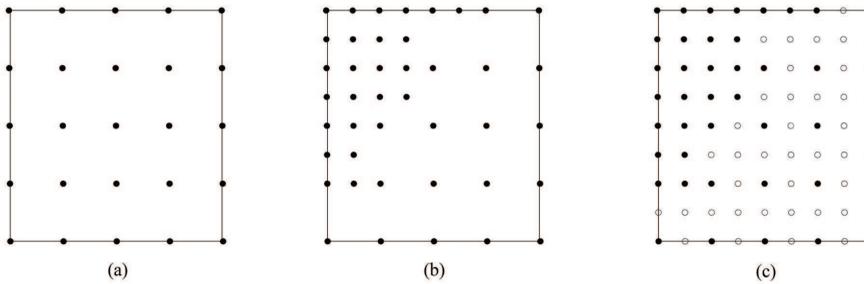


Figure 1: Schematic adaptation procedure and the node distribution, (a) coarse evenly spaced node distribution, (b) adaptive collocation node distribution, (c) adaptive collocation and base node distribution

without losing important information. On the other hand, high level wavelet coefficients that appear in the localized region cannot be neglected and the detailed local information should be included in the solution.

In the wavelet based adaptive scheme, an initial PDE solution is calculated on a coarse evenly spaced collocation nodes (Figure 1-a). The values of the solution at the collocation nodes are used to compute the wavelet coefficients at the next finer level of resolution. The node distribution is then refined only in the regions where the wavelet coefficients are greater than a prescribed threshold ϵ . The schematic figure of adaptive nodes is shown in Figure 1-b. In the next steps of the adaptation procedure where the collocation nodes are not evenly distributed in the domain, a slight revision is necessary. That is, the wavelet coefficients are calculated on the base nodes instead of the collocation nodes. The base node at each step of adaptation is defined as the coarsest evenly spaced distribution that includes all collocation nodes (see Figure 1-c). The PDE solution is first approximated on the base node and then the approximated values are used to compute the wavelet coefficients. The rest of the procedure remains unchanged. The adaptation procedure continues until the required level of accuracy is obtained.

The advantage of the adaptive wavelet scheme in comparison to the conventional schemes is that the wavelet coefficients can be used to detect those regions with localized features and are simply computed by the fast DWT. In contrast, the other automatic adaptive schemes previously reviewed are usually based upon the posterior error indicator in which the computation of the posterior indicator often dramatically penalizes simulation speed. From the computational point of view, the fast DWT algorithm requires only $O(N)$ operations, where N is the number of nodes in the base distribution. Another feature of the proposed wavelet based adaptive

scheme is its ability to create a non-uniform node distribution starting from an initial coarse uniform one. Moreover, the proposed adaptive scheme allows for an anisotropic node refinement in the vertical, horizontal or diagonal directions.

4 Numerical results

In this section, we show some numerical examples that demonstrate the efficiency of the proposed wavelet adaptive scheme in RBF collocation for nearly singular potential problems. In all the examples presented, we have used the MQ-RBFs for approximating the solution of the governing PDEs. For the test problems, we solve the Laplace, Poisson and Helmholtz potential equations with a near singularity in the domain or near the boundaries subject to the Dirichlet boundary condition. We have used the improved truncated-singular value decomposition scheme to find the expansion coefficients described in Emdadi, Kansa, Libre, Rahimian, and Shekarchi (2008) and Libre, Emdadi, Kansa, Rahimian, and Shekarchi (2008) as the condition number of the linear equation system becomes large. In each case, a comparison between the adaptive and fixed scheme is presented. Two types of norms were used to measure the error of approximation. The L_∞ and RMS as described below:

$$L_\infty = \max |u^{ex}(x_i) - u(x_i)| \tag{10}$$

$$RMS = \sqrt{1/N \sum |u^{ex}(x_i) - u(x_i)|^2} \tag{11}$$

where $u(x)$ is the approximate solution, $u^{ex}(x)$ is the exact solution, and N is the total number of collocation nodes. The results of the adaptive distribution are compared with those obtained on fixed distributions. In both the fixed and adapted distribution calculations, we have used a same number of nodes. The shape parameter was selected as $c=1/10$ and was kept constant for all nodes. The wavelet threshold parameter was selected as $\epsilon = 10^{-3}$ in all cases.

Example 1. Consider the following potential boundary value problem in the rectangle domain $\Omega = [0, 1]^2$.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2a^2(1+t^2)sech^2(a(v+\mu x-y))tanh(a(v+\mu x-y)) \quad (x,y) \in \Omega \tag{12a}$$

$$u = tanh(a(y - (v + \mu x))) \quad (x,y) \in \partial\Omega \tag{12b}$$

The analytical solution is given by

$$u^{ex}(x,y) = tanh(a(y - (v + \mu x)))$$

Here we consider the case where $a=15$, $v=0.4$ and $\mu=0.2$. The profile of $u(x,y)$ is shown in Figure 2. The first example represents the localized feature along the line $l(x,y) = v + \mu x - y$ that should be well suited to capture the near singularity. In doing so, the adaptive scheme should allocate more nodes to those parts of domain that exhibit sharp gradients. The adaptation procedure starts from a coarse base node distribution that contains $13 \times 13 = 169$ evenly spaced nodes and then successively refines the nodes where required. Figure 2 shows how the adaptive schemes distribute the nodes near the steep gradient. The numerical value of L_∞ and RMS norm errors of the first example at each level of resolution are shown in Table 1.

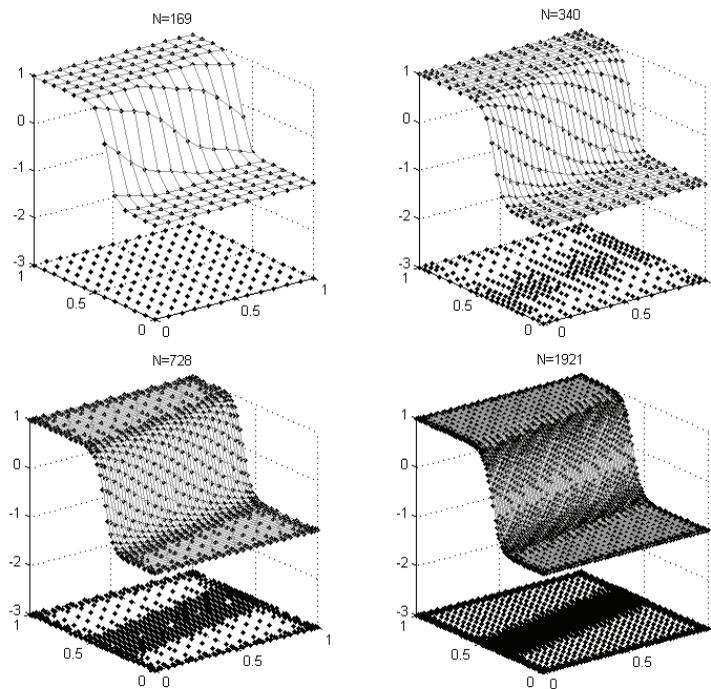


Figure 2: The base and adapted node distribution at different level of resolution. Example 1

The convergence error of the proposed adaptive scheme and the conventional fixed scheme are shown in Figure 3. After applying four steps of adaptation, the RMS error became less than $3.00E-4$ while the total number of collocation nodes is $N=1921$. To obtain the same accuracy in the fixed node distribution, $RMS \leq 3.00E-4$, we need $N=3249$ (57×57) nodes that are evenly distributed in the domain. The efficiency of adaptive versus the fixed scheme may be quantitatively measured by

Table 1: Error norms at the each step of the adaptive scheme, Example 1

Level	N	L_∞	RMS
1	169	3.62E-02	1.93E-02
2	340	1.32E-02	5.40E-03
3	728	4.40E-03	1.49E-03
4	1921	7.12E-04	2.96E-04

defining the compression index $I_c = N_a/N_u$, where N_a is the total number of adapted nodes and N_u is total number of nodes in the fixed distribution that is required to achieve the same accuracy level as the adaptive distribution. The lower the compression index, the more efficient is the adaptive algorithm. The $I_c=1$ indicates that there is no compression over the base fixed distribution while the $I_c \rightarrow 0$ shows the ability of adaptive scheme to remove the negligible nodes and compress the fine evenly spaced nodes. The compression index of the first example is $I_c=0.59$ for the RMS error $=3.00E-4$; this means the number of nodes of the adaptive distribution required to reach the RMS error $=3.00E-4$ is 59% of the fixed distribution.

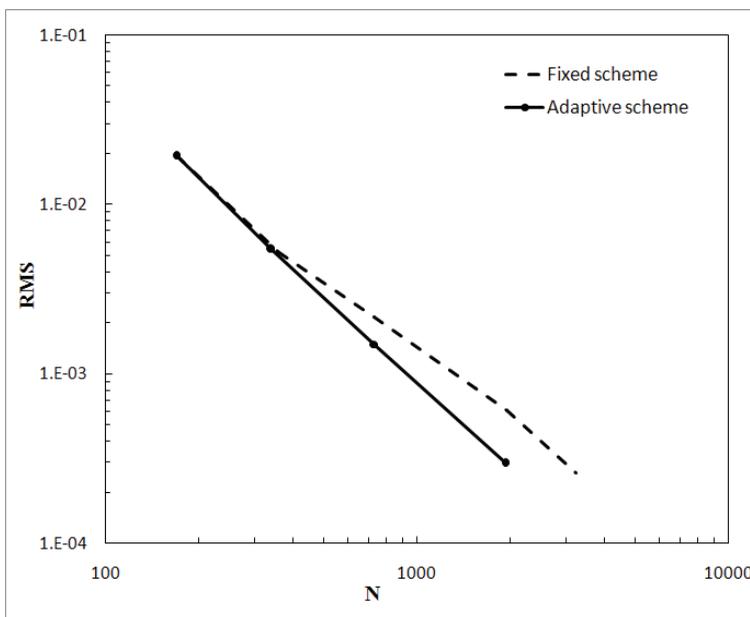


Figure 3: Error Convergence of adaptive scheme compared to fixed scheme, Example 1

Example 2. The second example is a typical boundary layer problem in potential PDEs. Consider the following Poisson type PDEs defined in rectangle domain $\Omega = [0, 1]^2$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (4m^2x^2 - 2m) \exp(-mx^2) + (4m^2y^2 - 2m) \exp(-my^2) \quad (x, y) \in \Omega \tag{13a}$$

subject to the following Dirichlet boundary condition

$$u = \exp(-mx^2) + \exp(-my^2) \quad (x, y) \in \partial\Omega \tag{13b}$$

We observe that there are two sharp regions near the boundaries $x=0$ and $y=0$. The localized features are sharpened by increasing the value of parameter m . In this example we consider the case with $m=200$. The node distributions must be refined near the boundaries with the sharp gradient so that the localized features can be captured properly. The refined node distribution shown in Figure 4 reveals that the proposed adaptive scheme is able to detect the regions with a sharp gradient and allocates more nodes where deemed necessary. At the fourth step of the adaptation procedure, total number of collocation points and the corresponding RMS norm are $N=1823$ and $RMS=4.29E-4$, respectively (see Table 2).

Table 2: Error norms at the each step of the adaptive scheme, Example 2

Level	N	L_∞	RMS
1	169	1.27E-01	4.98E-02
2	280	5.01E-02	1.69E-02
3	558	1.31E-02	4.61E-03
4	1823	1.06E-03	4.29E-04

Figure 5 shows the RMS norm versus the number of collocation nodes in both the adaptive and fixed schemes. Considering the vertical lines in the Figure 5, the conventional fixed scheme will result in $RMS=1.62E-3$ when $N=1823$. This means while the total number of collocation points was kept constant at $N=1823$, the adaptive scheme improves accuracy of the solution up to about four orders of magnitude. However, a more in-depth understanding of efficiency of the adaptive scheme may be gained if one considers the horizontal lines on the graphs, instead of the vertical lines. If $RMS \leq 4.30E-4$ is the acceptable accuracy level, it is clear from the horizontal line in Figure 5 that the number of required collocation nodes in the fixed and adaptive scheme are $N=1823$ and $N=3249$, respectively. In other

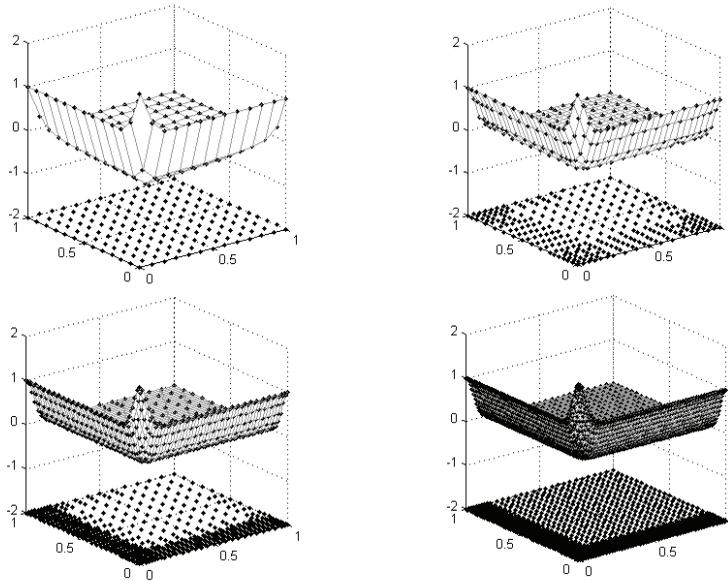


Figure 4: The base and adapted node distribution at different level of resolution. Example 2

words, the compression index is $I_c=56\%$ where the acceptable accuracy level is $RMS \leq 4.30E-4$. The reduced number of collocation nodes also reduces the condition number of coefficient matrix and mitigates the ill-conditioning problem. The total number of collocation nodes plays a crucial role in RBF analysis not only through the well-known ill-conditioning phenomenon in large scale problems, but also through the fact that more collocation nodes significantly increase the required memory and CPU time. For instance, the CPU time required for achieving the accuracy of $RMS \leq 4.30E-4$ using the proposed adaptive scheme is about 35 seconds. This computational time includes both the CPU time required for the adaptation procedure and the RBF solution procedure. On the other hand, the CPU time for achieving the accuracy level $RMS \leq 4.30E-4$ using the conventional fixed scheme is about 765 seconds that includes only the CPU time needed for the RBF solution procedure; the wavelet adaptive scheme is about 22 times faster. Therefore, the proposed adaptive scheme is well suited for nearly singular large scale problems, due to the significant savings of the memory and CPU time.

Example 3. The third example being investigated is the function with an internal point wise near singularity. The problem is to find a function $u(x,y)$, that satisfies

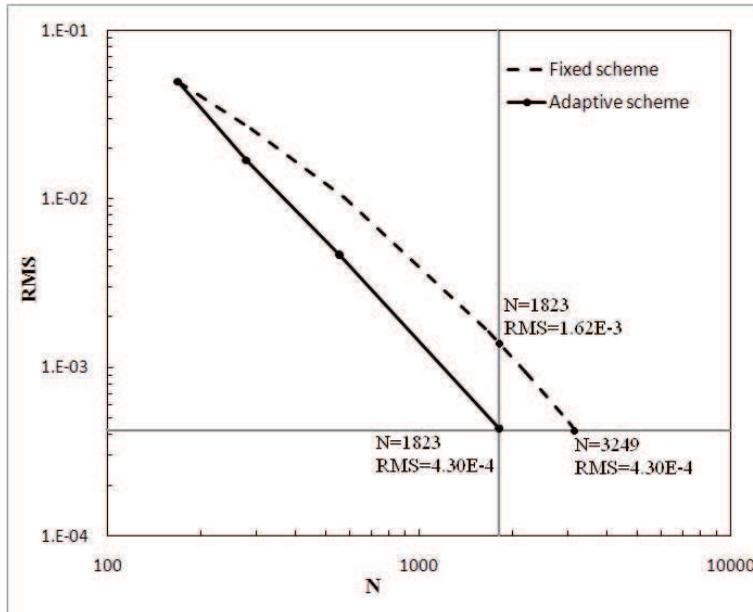


Figure 5: Error Convergence of adaptive scheme compared to fixed scheme, Example 2

the Poisson equation in a rectangle domain $\Omega = [0, 1]^2$.

$$\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 = 6(x+y)/m - (4/a^2 - 4(x-b)^2/a^4 - 4(y-b)^2/a^4) \exp(-(x-b)^2/a^2 - (y-b)^2/a^2) \quad (x,y) \in \Omega \quad (14a)$$

and satisfies the Dirichlet boundary condition

$$u = (x^3 + y^3)/m + \exp(-(x-b)^2/a^2 - (y-b)^2/a^2) \quad (x,y) \in \partial\Omega \quad (14b)$$

Note that the exact solution has a near singular point at $(x,y) = (b,b)$. Let us consider $m=2$, $a=0.05$ and $b=0.5$. The profile of the function and the nodes distribution at each step of adaptation procedure are shown in Figure 6. It is evident that the node distribution is dense in the center where the function has a strong point wise gradient as well as near the boundaries where the solution is prone to oscillations. The results clearly show that the adaptive procedure is able to detect the internal region with high spatial gradients and refine the distribution where required.

Applying the proposed adaptive scheme will result in $N=1507$ nodes and RMS error less than $8.50E-3$ after four steps of adaptation. The RMS and L_∞ error norms are

summarized in Table 3. The number of evenly spaced nodes required for achieving the $RMS \leq 8.50E-3$ in the conventional fixed scheme is $N=47 \times 47=2209$. So the compression index in the third problem is $I_c=1507/2209=0.68$.

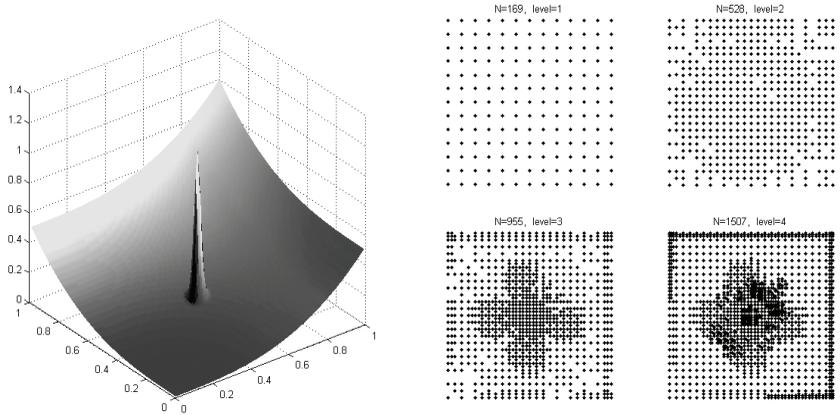


Figure 6: The exact solution and the base and adapted node distribution at different level of resolution. Example 3

Table 3: Error norms at the each step of the adaptive scheme, Example 3

Level	N	L_∞	RMS
1	169	1.27E-01	4.98E-02
2	280	5.01E-02	1.69E-02
3	558	1.31E-02	4.61E-03
4	1823	1.06E-03	4.29E-04

The examples presented in this work reveal that the proposed adaptive scheme is able to detect any type of near singularity near the boundary (e.g. example 2) or inside the domain (e.g. example 1 and 3) and refine the nodes where required. This is not the case with some other adaptive schemes that are able to detect only boundary singularities, see Hon (1999) and Ling and Trummer (2006). Another attractive feature of the proposed adaptive scheme is the automatic node clustering near the boundaries. It is well known that the largest errors in RBF methods occur near boundaries, see Libre, Emdadi, Kansa, Rahimian, and Shekarchi (2008) and Fornberg, Driscoll, Wright, and Charles (2002).. The reduction of the boundary error is usually performed by clustering the nodes near or slightly outside the boundary,

see Fedoseyev, Friedman, and Kansa (2002), or by using the pseudospectral methods, see Sarra (2005). The refined distributions of the third problem show that the proposed adaptive scheme automatically assigns more nodes to the boundary layer.

Example 4. In this example, the proposed adaptive scheme is generalized in order to obtain the near singular solutions of Helmholtz-type equations containing an approximate singularity both in the solution and its derivatives. Let the polar coordinate system (r, θ) be defined in the usual way with respect to the Cartesian coordinates $(x, y) = (r \cos \theta, r \sin \theta)$. The Helmholtz equation, written in polar coordinates takes the following form:

$$\partial^2 u / \partial r^2 + r^{-1} \partial u / \partial r + r^{-2} \partial^2 u / \partial \theta^2 + k^2 u = 0 \quad (x, y) \in \Omega \tag{15a}$$

Let $\Omega = [0, 1]^2$ be a rectangular domain, $k^2 = -1$ and consider the following Dirichlet boundary condition for the Helmholtz equation in polar coordinate system.

$$u(x, y) = r^{-1/2} \sinh(r) \cos(\theta/2) \quad (x, y) \in \partial\Omega \tag{15b}$$

The solution of this problem is weakly singular at the origin $(x, y) = (0, 0)$ but the limit of $r \rightarrow 0$, the solution exists in the vicinity of the origin.

$$u^{ex}(r, \theta) = r^{-1/2} \sinh(r) \cos(\theta/2)$$

$$\begin{cases} u(r=0, \theta) = \infty \\ \lim_{r \rightarrow 0} u(r, \theta) = 0 \end{cases} \quad \forall \theta \in \Omega$$

On the other hand, the derivative of the solution is strongly singular at the origin that means both the derivatives and its limit is singular at the origin.

$$\frac{\partial u}{\partial r} = \frac{\cosh(r) \cos(\theta)}{r^{1/2}} - \frac{\sinh(r) \cos(\theta)}{2r^{3/2}}$$

$$\begin{cases} \frac{\partial u(r=0, \theta)}{\partial r} = \infty \\ \lim_{r \rightarrow 0} \frac{\partial u}{\partial r} = \infty \end{cases} \quad \forall \theta \in \Omega$$

The profiles of the exact solution and its derivative are depicted in Figure 7. The adaptation procedure is applied to this example, but in contrast to the previous examples, no significant difference between the adaptive and fixed scheme is observed. The convergence rate observed in both the adaptive and fixed scheme is almost the same. This is due to the fact that the solution $u(r, \theta)$ is smooth enough in the vicinity of the origin and the adaptation procedure has little effect on the overall accuracy. On the other hand, the derivative of the solution is strongly singular at the origin but the proposed adaptive scheme can not detect the singularity

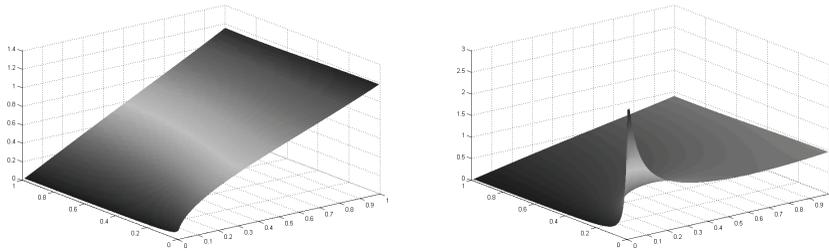


Figure 7: The exact solution (left) and its derivative (right), Example 4

in the solution derivative. Therefore, the adaptation procedure must be generalized in order that the adaptation scheme can detect the singularity in the derivatives. For this goal, the wavelet coefficients in the adaptation procedure are simply computed using the values of derivative of the solution at each level of resolution, instead of values of the solution. In this way, the adaptive scheme will be able to detect the singularity of the solution derivative. The generalized adaptive scheme is applied to this example and the adaptive node distributions are shown in Figure 8. The figure clearly shows that the generalized adaptive scheme is able to detect the singularity in the derivatives of the solution and refine the nodes where the derivatives exhibit sharp gradients.

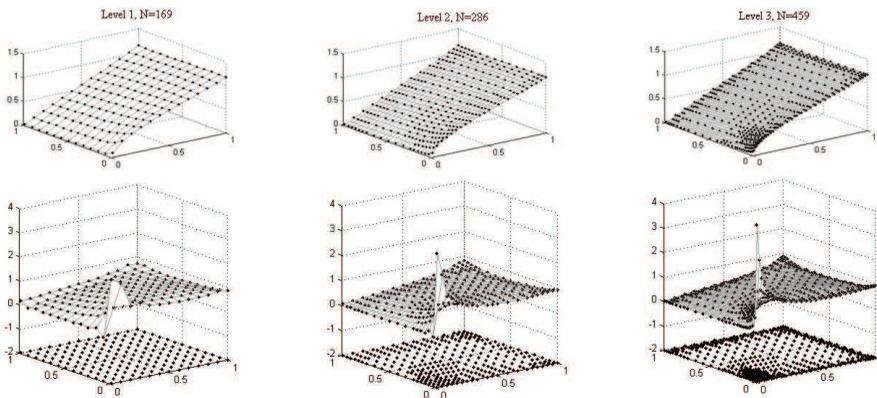


Figure 8: The solution of fourth example and its derivative at different levels of resolution

The L_∞ and RMS error norms are computed using the function derivatives and the results are summarized in Table 4. The adaptation procedure starts from a coarse-based mesh containing 169 nodes that are evenly distributed in the domain and after

Table 4: Error norms of function derivatives at the each step of the adaptive scheme, Example 4

Level	N	L_∞	RMS
1	169	1.34E-01	1.26E-01
2	286	9.37E-02	7.57E-02
3	459	7.69E-02	4.31E-02

three steps, there are 459 nodes primarily distributed around the origin. The convergence error of the solution derivative, $\partial u/\partial r$, using the adaptive scheme and fixed schemes are shown in Figure 9. The RMS error at the third step is $RMS=4.31E-02$. The conventional fixed scheme requires $N=28 \times 28=784$ nodes to obtain the same level of accuracy. That means the compression index in the fourth example is $I_c=0.59$. The fourth example clearly shows the ability of the proposed adaptive scheme to capture the singularity both in the solution and its derivatives.

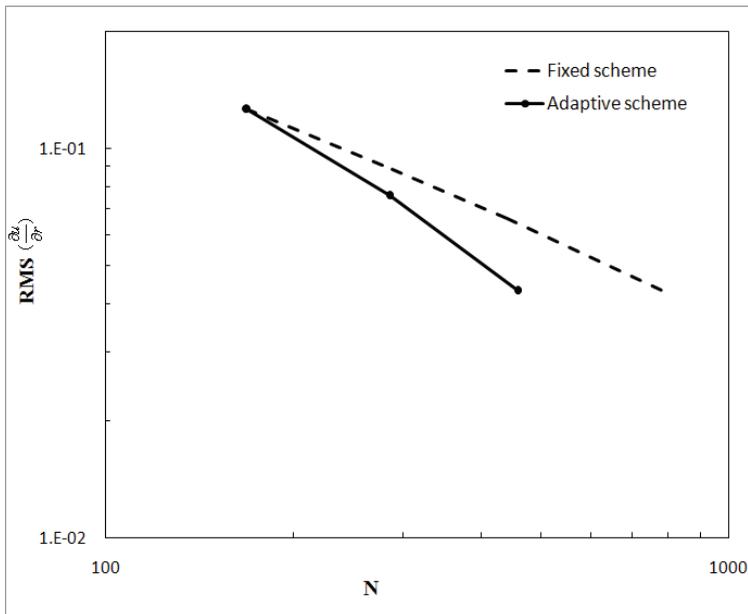


Figure 9: Error Convergence of derivative of the solution using adaptive scheme compared to fixed scheme, Example 4

5 Conclusion

In this paper, we have shown that the adaptive wavelet scheme is able to detect the localized features of PDE solutions in the domain or near the boundary and to allocate more nodes to the essential regions. In the framework of the adaptation scheme, the wavelet coefficients were used as the parameters that indicate the accuracy of the solution and determine where the node distribution can be coarsened or refined. Therefore, the number of collocation nodes is optimized without deteriorating the accuracy of the solution. Certain aspects of the convergence of the proposed adaptive wavelet scheme have been discussed through numerical examples. It has been shown that the proposed adaptive scheme is able to detect the near singularity both in the function and its derivatives. The examples investigated reveal that this method achieves very good performance in terms of convergence rate and CPU time saving. The adaptation algorithm complexity $O(N)$ is considerably lower than the computational cost of other automatic adaptation strategies proposed in the literature. The wavelet based adaptive scheme is a relatively new technique, still under study and development within different areas of mathematics and physics. Work is in progress to generalize the proposed adaptive scheme both to nonlinear and time dependent PDEs.

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